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The Dynamics of Capital Accumulation in the US: Simulations after Piketty[1]

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Abstract

We develop a dynamic model where a competitive firm produces a single good from labor and capital, with market clearing rates of return. Individuals are heterogeneous in skills, with an endowment in capital/wealth increasing in skill. Individuals aspire to a standard consumption level, with a constant marginal propensity to consume out of income above this level. We define a steady state of this model as an equilibrium where factor returns and wealth shares remain constant. We calibrate the model to the US economy and obtain that a steady state exists. We then study three variants of the model: one with a higher rate of return for large capitals than for smaller ones, one with social mobility, and one with a capital levy financing a lump sum transfer. In all variants, a steady state exists. We also run the model starting from the 2012 US wealth distribution and obtain convergence to the steady state in the basic model as well as in all variants. Convergence takes a long time and is non monotone, with factor returns and wealth shares moving away from their steady state values for long periods.

Keywords: Piketty, dynamics of wealth accumulation, convergence to steady state, spirit of capitalism, differential rates of return to capital, intergenerational mobility, capital levy, US calibration.

JEL Codes: D31, D58, E37

1 Introduction

The recent book by Piketty (2014) has rekindled an interest in understanding why wealth is so concentrated at the top of the distribution, an evident phenomenon in several major economies, and especially in the US.

That wealth distributions are skewed to the right and display thick upper tails has been known since Pareto's "Cours d'Economie Politique" (1909). Benhabib and Bisin (2016) survey the early literature developed to understand the mechanisms at play in generating thick-tailed wealth distributions. All these early attempts were mechanical and lacked economic microfoundations. Beginning in the 1990s, economists have started to build microfounded models designed to understand the determinants of the properties of wealth distributions. Quadrini and Ríos-Rull (1997) survey this literature, where the models reviewed are heterogeneous agent versions of standard neoclassical growth models. Two types of model have been developed, *dynastic* and *life-cycle*. In their words (p 23), "The dynastic model includes the infinitely lived agent abstraction and assumes that people care for their descendants as if they were themselves, and the life cycle model includes overlapping generations of finitely lived agents who do not care about their descendants. Thus, the main motive for saving (...) differs in these two types of models: in dynastic models, people save to improve their descendants' consumption, while in life cycle models, people save to improve their own consumption during retirement." In both cases, preferences are represented by the discounted sum of a per period utility function.

Both types of model focus on the steady state (where variables grow at a constant rate –perhaps zero– over time) and have trouble reproducing the actual wealth distributions.[1] The simplest versions of these models assume that agents' earnings are deterministic. In the absence of shocks, there is no need for precautionary savings in the dynastic models, so that the steady state wealth distribution inherits mechanically all the properties of the initial distribution. As for the deterministic life cycle models, Huggett (1996) shows that calibrated models generate aggregate savings that are too low to explain the US savings rate, and too little wealth concentration in the upper tail.

The next step undertaken by the literature has consisted in adding uninsurable idiosyncratic shocks to labor earnings. The seminal papers have adopted the dynastic approach and are due to Bewley (1983) and Aiya-

[1] Most work has been done on the US case.

gari (1994). In these models, where agents save for precautionary reasons to smooth consumption, the key determinant of the wealth distribution is the volatility of individual earnings, not permanent differences in earnings across households (Constantinides and Duffie 1996). The literature has also added uninsurable labor shocks to life cycle models. Huggett (1996) further adds uninsurable lifetime uncertainty, generating both precautionary savings and accidental bequests. In all cases, models under-estimate the fraction of wealth accumulated at the top of the distribution. More recently, the literature has introduced stochastic returns for capital as well as labor. Benhabib *et al* (2011) obtain a stationary wealth distribution which is Pareto in its right tail, with this tail populated by dynasties which have realized a long streak of high rates of return on capital, so that capital income risk, rather than stochastic labor income, drives the properties of this right tail.[2]

In this paper, we develop in Section 2 a deterministic dynastic model with infinitely lived agents who differ in skills. We first depart from the literature surveyed above in how we model individual preferences. Agents do not maximize a discounted infinite sum of consumption over time, but rather a single-period utility function of consumption and savings, repeated in each period. Moreover, we assume there is a socially expected standard of living for people in this society. It is not subsistence consumption, but rather the consumption level to which ordinary people aspire, which is generated by advertising and the media (in the US, this level would define a successful middle-class life).[3] The marginal propensity to consume out of income is assumed to be unity below this standard consumption level, and a constant lower than one above this level. One way to interpret these preferences is that agents desire to consume and to accumulate wealth for its own sake. They do so because 'money is life's report card,' as the caption of a New Yorker cartoon said. In this view, accumulation for its own sake is the motivation for most members of the wealthy class, for success in the game of life is judged by one's wealth. As noted by Cooper (1979), "Persons in the wealth category we are now discussing have more current income than they can expend. Beyond a certain point, the real value of greater wealth is power, control, and security". Max Weber (1905) has argued that private accumulation of wealth as an end

[2]See Quadrini (2000) for a dynastic model with stochastic returns to entrepreneurship under market imperfections and financial constraints, and Cagetti and De Nardi (2006) for a life cycle model building on Quadrini (2000).

[3]See Hubbard, Skinner, and Zeldes (1995) for a life cycle model with subsistence consumption.

in itself, rather than for consumption purpose –a behavior he dubbed the *spirit of capitalism*– has been the main driver of Industrial Revolution in Europe. Piketty and Zucman (2015, p1346) study a similar (although not equivalent) "Wealth-in-the-Utility-Function", where the utility function is defined over consumption and (the increase in) wealth, and justify the latter by referring to wealth as "a signal of their ability or virtue".[4] Our first objective is then to assess the consequences of assuming such preferences on the dynamics of capital accumulation in a model calibrated to reflect broadly the US economy.

We assume that the initial distribution of wealth is monotone increasing in an individual's skill. A firm, using a CES production function whose inputs are efficiency units of labor and capital, maximizes profits. Consumers-workers offer inelastically their entire endowment of skilled labor to the firm; they demand the consumption good and supply capital to the firm in order to maximize preferences described above. The interest rate and real wage equilibrate the markets for labor and capital. There are proportional taxes on capital and labor income, the revenues from which are returned as a demogrant to each worker. Skills are assumed to increase at an exogenous rate, as does the expected standard of living. The main fundamental changing over time is the distribution of capital/wealth.

The literature surveyed above concentrates on establishing the properties of the steady state of the economy (where real rates of return, capital output ratio and wealth shares remain constant over time). This approach is incomplete, as recognized by Benhabib *et al* (2015) who write "This comparison implicitly assumes that the wealth distribution for the U.S. is close to stationary. This might in general not be the case if the wealth distribution is hit frequently enough by aggregate shocks like wars, major business cycle events (e.g., a depression), changes in tax schemes, social insurance institutions, and so on; see Saez and Piketty (2003). We leave the study of the transition of the distribution of wealth for future work" (p 16). Our second objective is then to go beyond the study of the steady state equilibrium and to also investigate whether and how the model converges to this steady state when we start, at period zero, with the wealth distribution observed in the US in 2012 by Saez and Zucman (2016).

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[4] Our formulation generates a consumption pattern reflecting Saez and Zucman (2016)'s evidence of substantial saving rate differentials across wealth levels, so that, in Benhabib *et al* (2015)'s words, "the rich can get richer through savings, while the poor may not save enough to escape a poverty trap" (p 3).

Our third objective is to assess how the introduction of three relevant departures from our basic "vanilla" model affects both the steady state and the convergence to this state from the 2012 US wealth distribution. First, introducing heterogenous rates of returns is often mentioned as a promising way to move the models closer to explaining actual wealth data, and especially its concentration at the top (see for instance Quadrini and Ríos-Rull (1997), which mention that "the portfolio of wealthy households typically includes assets that yield higher returns than the assets of poorer households" (p 29), and Benhabib and Bisin (2016)). There is also increasing empirical support for this form of heterogeneity, even though Piketty and Zucman (2015) stress the poor quality of the data. Piketty (2014, chapter 2) estimates these rates of return using university endowments, which are public information in the United States because non-profit institutions must report these data. He obtains returns (over the 1980-2010 period) ranging from 6.2% for endowments less than \$100 million to 10.2% for endowments much larger than \$1 billion. Saez and Zucman (2016) find the same pattern for the universe of U.S. foundations, and Piketty and Zucman (2015, Table 15.1) using Forbes global wealth rankings. Saez and Zucman (2016, online appendix, Tables B29, B30, and B31) show mildly increasing pre-tax returns in wealth over the period 1980-2012. Administrative data from Scandinavian countries also show pre-tax returns increasing in wealth. Fagereng *et al* (2015, 2016) find returns significantly increasing in wealth only for high wealth classes, above the top 10%, in Norway. Bach *et al* (2015) find higher returns on large wealth portfolios for Sweden. In section 3, we construct a highly simplified model with one rate of return for capitals in the top 1% of the wealth distribution and a smaller rate of return for capitals in the bottom 99%. We continue to assume that the average rate of return clears the capital market.

In Section 4, we add intergenerational mobility to our original model. We take a generation to last for 50 years, and model this by assuming that each individual has a 2% probability of dying each year, upon which his capital passes down, without taxation, to his single offspring. The offspring's skill level – and hence her labor earnings – are not inherited, but are taken to be determined by the income intergenerational mobility matrix of Chetty *et al* (2014).[5]

Finally, in the spirit of Piketty (2014), we introduce in Section 5 a capital

[5] Kopczuk and Lupton (2007) study empirically bequests motives in the US and obtain that roughly three-fourths of the elderly single population in their sample has a bequest motive, which can best be described as egoistic, namely "a desire to have positive net

levy on the top wealth decile, whose proceeds are redistributed as a lump sum transfer.

In all variants of the model, we obtain that a steady state exists. We also obtain convergence to this steady state when we start with the 2012 US wealth distribution. The main results obtained with the basic version of the model as well as with each variant are summarized at the end of each section, and we recap our main results in Section 6.

2 The basic model

We present the model and solve for its equilibrium date by date in section 2.1, and we solve for its steady-state (where real rates of return, capital output ratio and wealth shares remain constant over time) in section 2.2. We then calibrate the model to the modern US economy in section 2.3 and describe our numerical results in section 2.4.

2.1 Presentation of the basic model

The economy consists of a continuum of individuals who differ in their skill level s. The distribution of skill levels is represented by the c.d.f. $F(s)$ over $[0, \infty[$, where

$$\bar{s} = \int_0^\infty s dF(s)$$

denotes the average skill. We use the subscript t to denote the date at which a variable is measured. The model starts at period $t = 0$, with a distribution of wealth denoted by $S_0(s)$. We assume that wealth $S_0(s)$ is monotone increasing in s, and that skills increase (exogenously) by a factor $(1 + g)$ per period.

There is a single good in the economy. Preferences are non-traditional. We assume there is a socially expected standard of living for people in this society, produced by a consumption level c_0 at date 0. This expected consumption level increases by a factor of $(1 + g)$ per period. We do not call this subsistence consumption – we set it at \$100,000 in the simulations. It is the consumption level to which ordinary people aspire, which is generated

worth upon death". They model this desire with a per-period utility which is an increasing function of both consumption (if alive) and bequest (if dead)–i.e., the counterpart of the "wealth in utility function" used in our model and in Piketty and Zucman (2015).

by advertising and the media (in the US, this level would define a successful middle-class life). A sufficiently wealthy individual at date t chooses her consumption c and investment I to maximize a Stone-Geary utility function as follows:

$$\begin{aligned} \max \quad & (c_t - c_0(1+g)^{t-1})^{\alpha} I_t^{1-\alpha} \\ & \text{subject to} \\ & c_t + I_t \leq y_t(s), \\ & c_t \geq c_0(1+g)^{t-1}, \end{aligned} \tag{1}$$

where $y_t(s)$ is the income of individual s at date t. If there is no solution to program (1), because income is insufficient to purchase the consumption level $c_0(1+g)^{t-1}$, then the individual consumes out of wealth. To be precise,

$$c_t(s) = \begin{cases} y_t(s) + S_{t-1}(s) \text{ if } y_t(s) + S_{t-1}(s) \leq (1+g)^{t-1}c_0 \text{ (case 1)} \\ (1+g)^{t-1}c_0 \text{ if } y_t(s) \leq (1+g)^{t-1}c_0 \leq y_t(s) + S_{t-1}(s) \text{ (case 2)} \\ (1+g)^{t-1}c_0 + \alpha\left(y_t(s) - (1+g)^{t-1}c_0\right) \text{ if } y_t(s) > (1+g)^{t-1}c_0 \text{ (case 3)} \end{cases} \tag{2}$$

In case 1, the individual consumes his income plus his wealth $S_{t-1}(s)$, and those together do not suffice to generate the socially expected consumption of $c_0(1+g)^{t-1}$. In case 2, when her income does not suffice to allow socially expected consumption but her total asset position does, she consumes exactly the socially acceptable consumption level. In case 3, where her income alone suffices to allow socially acceptable consumption, she solves program (1) with α the marginal propensity to consume out of income. Thus, investment is given by

$$I_t(s) = \begin{cases} -S_{t-1}(s) \leq 0 \text{ if case 1} \\ y_t(s) - c_0(1+g)^{t-1} \leq 0 \text{ if case 2} \\ y_t(s) - c_t(s) > 0 \text{ if case 3} \end{cases} \tag{3}$$

The dynamics of wealth are given by

$$S_t(s) = S_{t-1}(s) + I_t(s). \tag{4}$$

We now turn to the production side of the economy. A single firm produces the consumption good, using a CES technology given by

$$y(K, L) = A\left(aK^{\frac{\delta-1}{\delta}} + (1-a)L^{\frac{\delta-1}{\delta}}\right)^{\frac{\delta}{\delta-1}} \tag{5}$$

where y, K and L are per capita income, capital, and labor in efficiency units. The only technical change in the model is induced by the exogenous increase in labor skills. The firm faces an interest rate r and a real wage per efficiency unit of labor w, and maximizes profits. We denote by d the annual rate of capital depreciation.

The firm's profit is defined as

$$y(K_t, L_t) - w_t L_t - (r_t + d)K_t,$$

whose differentiation gives the following FOC for the demands for labor

$$w_t = (1-a)y_t^{1/\delta} L_t^{-1/\delta} A^{\frac{\delta-1}{\delta}} \tag{6}$$

and capital

$$r_t + d = a y_t^{1/\delta} K_t^{-1/\delta} A^{\frac{\delta-1}{\delta}}. \tag{7}$$

Note that the firm replaces depreciated capital from income, so that the investor can cash out his entire capital stock at the end of the period, which explains why the depreciation does not appear in equation (4).

The market clearing equations are

$$K_t = \int_0^\infty S_{t-1}(s) dF(s), \tag{8}$$

$$L_t = \int_0^\infty (1+g)^{t-1} s dF(s) = (1+g)^{t-1}\bar{s}. \tag{9}$$

Using (5) and (9) in (6) and (7), the FOCs with respect to L and K simplify to, respectively,

$$w_t = (1-a)A\left(a\left(\frac{K_t}{(1+g)^{t-1}\bar{s}}\right)^{\frac{\delta-1}{\delta}} + 1 - a\right)^{\frac{1}{\delta-1}} \tag{10}$$

and

$$K_t = (1+g)^{t-1}\bar{s}\left(\frac{\left(\frac{r_t+d}{aA}\right)^{\delta-1} - a}{1-a}\right)^{\frac{\delta}{1-\delta}}. \tag{11}$$

Finally, we assume an exogenously given income tax rate τ, the revenues from which are returned to citizens as a demogrant. Thus income for an agent of type s in year t is given by

$$y_t(s) = (1-\tau)\left(w_t s(1+g)^{t-1} + r_t S_{t-1}(s)\right) + \tau\left(w_t \bar{s}(1+g)^{t-1} + r_t K_t\right). \tag{12}$$

2.2 The steady-state

We now define the steady state of this economy.

Define first

$$s_1(t) = \sup_s \{s \,| S_t(s) = 0\}$$

as the highest skill level with no wealth at time t.

Definition 1 *A steady-state of the basic model is an equilibrium of the model defined in section 2.1, where $r_t = r$, $w_t = w$, $s_1(t) = s_1$, for all t, and where the variables $y_t(s)$, $c_t(s)$, $I_t(s)$, K_t, L_t, $S_t(s)$ all grow at rate g.*

We can then represent

$$\begin{aligned} y_t(s) &= (1+g)^{t-1} y^*(s),\ t \geq 1 \\ S_t(s) &= (1+g)^{t-1} S^*(s),\ t \geq 0 \\ K_t &= (1+g)^{t-1} K^*,\ t \geq 1 \\ L_t &= (1+g)^{t-1} \bar{s},\ t \geq 1. \end{aligned}$$

From the FOC (11) with respect to K and the definition of K^*, we obtain

$$K^* = \bar{s} \left(\frac{\left(\frac{r_t+d}{aA}\right)^{\delta-1} - a}{1-a} \right)^{\frac{\delta}{1-\delta}}. \tag{13}$$

Solving for r, we obtain

$$r = \left(\left(\frac{K^*}{\bar{s}} \right)^{\frac{1-\delta}{\delta}} (1-a) + a \right)^{\frac{1}{\delta-1}} aA - d. \tag{14}$$

Similarly, from the FOC (10) with respect to L , we obtain

$$w = (1-a)A \left(a \left(\frac{K^*}{\bar{s}} \right)^{\frac{\delta-1}{\delta}} + 1 - a \right)^{\frac{1}{\delta-1}}. \tag{15}$$

By definition of s_1,

$$y^*(s) = (1-\tau) ws + \tau(w\bar{s} + rK^*) \leq c_0 \text{ for all } s \leq s_1.$$

We can then define

$$s_1 = \frac{c_0 - \tau(w\bar{s} + rK^*)}{(1-\tau)w}. \tag{16}$$

Observe that there cannot be case 2 agents (see (2)) at the steady state – i.e., agents with positive wealth but negative investments– since their wealth would decrease at every date, contradicting the definition of a steady state. Hence, all those with $s > s_1$ do invest and are such that

$$S_t(s) = S_{t-1}(s) + (1-\alpha)\left(y_t(s) - c_0(1+g)^{t-1}\right).$$

Dividing by $(1+g)^{t-1}$ gives the steady state

$$S^*(s) = \frac{S^*(s)}{1+g} + (1-\alpha)(y^*(s) - c_0) \tag{17}$$

which solves to

$$S^*(s) = \frac{1+g}{g}(1-\alpha)(y^*(s) - c_0).$$

Therefore,

$$S^*(s) = \frac{(1+g)(1-\alpha)}{g}\left((1-\tau)\left(ws + \frac{r}{1+g}S^*(s)\right) + \tau\left(w\bar{s} + rK^*\right) - c_0\right) \text{ for } s > s_1,$$

which solves to

$$\frac{S^*(s)}{1+g} = \frac{(1-\tau)ws + \tau(w\bar{s} + rK^*) - c_0}{\gamma} \text{ for } s > s_1, \tag{18}$$

where

$$\gamma = \frac{g}{1-\alpha} - r(1-\tau).$$

Using (8) together with (18), we obtain:

$$\left(\gamma - \tau r\left(1 - F(s_1)\right)\right)K^* = (1-\tau)w\int_{s_1}^{\infty} s dF(s) + (1-F(s_1))(\tau w\bar{s} - c_0). \tag{19}$$

Recall that the lower bound of the integral on the RHS, s_1, depends on K^* (see equation (16)). Existence of the steady state depends on the existence of a solution to the equation (19), an equation in the single unknown K^*. Observe also from equation (18) that the distribution of wealth at equilibrium is linear in s for $s > s_1$.

If a steady state equilibrium exists, then we obtain by definition that the wealth of every s grows at rate $(1+g)$. (This is obviously true for those with wealth zero.) Therefore, the fraction of total wealth owned by any sub-class of the population is constant.

2.3 Calibration

We start with the calibration of the production function. Recall that the production function has the CES form given by (5). We choose $\delta = 0.85$, and we stress at the end of section 2.4 that our results are not affected as we increase δ from 0.85 to 1.25. One period is deemed to be one calendar year. The capital income ratio is 4.5 in the U.S. Depreciation is about 10% of GNP, which suggests a rate of depreciation $d = 0.02$. We assume that $g = 0.02$. The distribution of skills is taken to be lognormal. The unit of skill has no meaning: we take the median skill level to be 0.85 and the mean $\bar{s}$ to be 1. We then have that $L = 1$ at period 1. We develop in Annex 1 how we calibrate the production function.

As for preferences, we choose $c_0 = 100$ and $\alpha = 0.6$, based on the fact that the propensity to consume for the wealthy is about 0.6 out of income. There are many estimates of the marginal propensity to consume of the wealthy (see Carroll *et al* (2014)), which include 0.6. We chose this value as it generates an aggregate savings rate in our models of about 9%, conforming roughly with reality.[6]

Finally, the taxation rate τ is set at 0.35 throughout the paper.

2.4 Numerical Results: Steady state and convergence

A steady state exists for the parameter values detailed in section 2.3. The main characteristics of the steady state are described in Table 1.

r	6.32%
w	\$67,750
$K^*/y(K^*, \bar{s})$	4.45
$F(s_1)$	83.8%

Table 1: Steady state allocation

Observe first that $r(1-\tau) = 4.11\%$, so that $r(1-\tau) > g$ is consistent with constant wealth shares in the steady state. The value of the capital output ratio is, at 4.45, very close to the targeted value of 4.5 used to calibrate the parameters of the production function (see Annex 1). Also, Piketty (2014)

[6] According to US census data, the savings rate has varied between 5% and 15% over the last forty years.

computes that the share of capital income, net of depreciation, in GNP is 28% (see Annex 1), from which we infer that $r = 6.22\%$, which is very close to our steady state value of 6.32%.[7] The saving rate of the economy in steady state is then

$$\begin{aligned} g\frac{S^*}{y(K^*, \bar{s})} &= (1+g)g\frac{S^*}{(1+g)y(K^*, \bar{s})} \\ &= (1+g)g\frac{K^*}{y(K^*, \bar{s})} \\ &= 0.091 \end{aligned}$$

As for the wealth distribution, 84% of individuals have no wealth at the steady state equilibrium, with wealth increasing linearly in skill for the top 16% of individuals. The steady state distribution of wealth is summarized in Table 2, where we compare it with the actual wealth distribution taken from Saez and Zucman (2016, Appendix Table B1).

Group	Wealth share in	
	steady state	actual
bottom 50%	0	0
top 10%	92%	77.2%
top 5%	68.7%	64.6%
top 1%	25.4%	41.8%
top 0.5%	15.4%	34.5%
top 0.1%	4.5%	22%
top 0.01%	0.7%	11.2%

Table 2: Wealth shares at steady state equilibrium and in Saez-Zucman (2016)

The steady state equilibrium nearly reproduces the wealth shares accruing to the bottom 50% and to the top 5%, but under-estimates both the share going to the "patrimonial middle-class" (from 5th to 9th decile) and, especially, to the very top (1% to 0.01%) of the wealth distribution. Observe also that the under-estimation increases as one focuses on the very top of the distribution, since the actual wealth share of the top 1% is 1.64 times higher in reality than in the computed steady state, with this ratio increasing to 2.24 for the top 0.5%, 4.89 for the top 0.1% and 16 for the top 0.01%.

[7] The share of capital income, including depreciation, in GNP is 37% in our steady state.

The rapidity with which the actual wealth share at the top of the distribution has increased in recent years is much greater than occurs in simulations of convergence to the steady state. We attribute this phenomenon to imperfections in competition that we have not modelled– for instance, the assault upon unionization in the US since the Reagan presidency and global factors, such as the rise of China.

If this analysis is correct, then the fact that wealth is becoming more concentrated in reality is either due to a process of convergence to the steady state, or to a departure of reality from the model. Our next step is then to check the convergence to the steady equilibrium, starting from the Saez and Zucman (2016)'s distribution. More precisely, we assume that the initial distribution $S_0(s)$ is linear by parts over s, with 7 different brackets reproducing Saez and Zucman (2016)'s top brackets. Total capital per worker at the beginning of period 1 is set at 4.5 times \$108,300 (see section 2.3).

We run the model for 500 periods, as follows. For each period, we begin with the wealth function $S_{t-1}(s)$ at the beginning of date t. K_t and L_t are determined by (8) and (9). Equations (10) and (11) determine w_t and r_t. Income y_t is determined by (12). Consumption and investment are determined by (2) and (3). $S_t(s)$ is determined by (4). The next iteration begins.

The model converges to the steady state equilibrium, with a very interesting convergence pattern. The equilibrium interest rate is lower at $t = 1$ (r_1 =6.19%) than its steady value, and increases for the first 48 periods, to reach a maximum of 6.69% (see Figure 1).[8] It then decreases and converges to the steady state value of 6.32%. A similar pattern of overshooting also occurs for the equilibrium wage rate (see Figure 2), which starts at $t = 1$ at a higher level (\$68,408) than the steady state, decreases for 48 periods to reach a minimum of \$66,055, and then increases and converges to the steady state level of \$67,750. In both cases, the equilibrium rate of return remains quite close to its steady state value, with a maximum gap (reached in both cases at $t = 48$) of 5.75% for r and 2.5% for w.

Insert Figures 1 and 2 around here

The capital output ratio behaves very similarly to the equilibrium wage, as can be seen in Figure 3. It starts above its steady state value, reaches its

[8]In all figures illustrating convergence, the horizontal line is set at the steady state level of the variable depicted.

minimum amount of 4.3 after 48 periods, and then increases to its steady state value. At any point in time its value is within 3.7% of its steady state value of 4.45.

Insert Figure 3 around here

We now move to the evolution of the wealth shares. The wealth share of the patrimonial middle class (from 5th to 9th decile) decreases with time, with a rate of decrease higher in the first periods, and a minimum attained after 394 periods (see Figure 4). Here also, we observe a slight overshooting, since the equilibrium share after 394 periods is, at 7.79%, lower than its steady state value of 8.01%.

The other wealth shares we study do not exhibit a pattern of overshooting, with the wealth shares of both the top 1% and top 0.1% first increasing (for 29 and 13 periods, respectively) and then decreasing and converging to their steady state values (see Figures 5 and 6).

Insert Figures 4 to 6 around here

Here are the main conclusions we can draw from these results. First, we observe convergence to the steady state, with this convergence happening more quickly for r, w and the capital output ratio, and more slowly for wealth shares. Second, during convergence, we may observe overshooting, with a variable crossing its steady state value. Third, and perhaps more important, the evolution of several variables is not monotone with time. For instance, even though the wealth shares of the top 1% and top 0.1% are above their steady states values at the beginning of time, they keep on increasing for, respectively, 29 and 13 years before peaking up and then starting their downward convergence to their steady state levels. The main message of this section is then that, even when convergence occurs, the non-monotonicity of wealth shares (and other variables such as interest rate, wage or capital output ratio) makes the task of the econometrician extremely difficult, since even the observation of several decades of increasing wealth shares does not mean that we are converging to a higher steady state, but rather the opposite.

Finally, we have studied the sensitivity of our results to the value taken by two parameters: the elasticity of substitution δ and the standard of living level c_0. The impact of the elasticity of substitution δ on the results is inconsequential, as the steady state values of w, r, K/y and of the various wealth shares remain practically constant as we vary δ from 0.85 to 1.25 (see Table A2.1 in Annex 2). This shows that the debate about the value of δ, following Piketty (2014), is somewhat moot, at least for our formulation of the economy.

This is in stark contrast with the very large impact of the expected standard of living c_0 on the steady state results, as reported in Table 3.

					Wealth shares			
c_0	w	r	K/y	$w\bar{s}/y$	0-50	50-90	90-99	top 1%
\$100,000	\$67,750	6.32%	4.45	0.629	0	8.01%	66.63%	25.36%
\$85,000	\$75,971	4.82%	5.28	0.640	0	40.23%	47.17%	12.6%
\$70,000	\$93,281	2.66%	7.30	0.660	12.57%	52.83%	28.32%	6.27%
\$55,000	\$108,034	1.47%	9.38	0.675	25.06%	48.94%	21.59%	4.41%

Table 3: Steady state results as a function of c_0

As the standard of living c_0 decreases from \$100,000 to \$55,000, the middle class starts accumulating capital, resulting in a more-than-doubling of the capital/output ratio. As a consequence, the equilibrium rate of return of capital decreases while the equilibrium wage increases. The labor share of income in GNP ($w\bar{s}/y$) increases, although in a less spectacular fashion than w. The impact of a lower c_0 on wealth shares is tremendous, with a quarter of capital accruing at steady state to the bottom half of the distribution, one half to the patrimonial middle class, and with the share of the top 1% decreasing by a factor five!

Indeed the dramatic results from decreasing c_0 in the model would seem to support the views, of those like Skildesky and Skildesky (2012), who see a solution to capitalism's ills in the reduction of consumption. We do not believe, however, that a mass reduction in desired consumption levels is sociologically feasible or desirable. The steady-state savings rate with $c_0 = \$55,000$ is 19%, which seems quite incompatible with the consumerist nature of American capitalism.

We now introduce rates of return which vary with the amount of individual capital invested.

3 Differential rates of return

Large capitals earn significantly higher rates of return than small ones, as shown with data from US university endowments (Piketty 2014), US foundations (Saez and Zucman 2014), Forbes global wealth rankings (Piketty and Zucman 2015), or administrative data from Scandinavian countries (see Fagereng *et al* (2015, 2016) for Norway and Bach *et al* (2015) for Sweden).

In this section, we ask whether steady states continue to exist with differential rates of return. To do so, we make a very simple assumption, that the rate of return is some number r_1 for wealths accruing to those individuals in the bottom 99% of the wealth distribution, and kr_1 for those in the top 1%, where k is a parameter greater than one.[9] We assume that the average rate of return, r, continues to clear the capital market.

Consequently, in the steady state, if one exists, we must have

$$rK^* = r_1 \int_{s_1}^{q99} \frac{S^*(s)dF(s)}{1+g} + kr_1 \int_{q99}^{\infty} \frac{S^*(s)dF(s)}{1+g}, \tag{20}$$

where $q99$ is the 99th centile of the distribution F, which is also the 99th centile of the wealth distribution in our model. The equations derived in section 2.2 for r, K^*, w and s_1 remain identical. However, equation (18) for $S^*(s)$ now bifurcates into

$$\frac{S^*(s)}{1+g} = \frac{(1-\tau)ws + \tau(w\bar{s} + rK^*) - c_0}{\gamma(s)} \text{ for } s > s_1, \tag{21}$$

where

$$\gamma(s) = \begin{cases} \gamma_1 = \frac{g}{1-\alpha} - r_1(1-\tau) \text{ if } s_1 < s < q^{99}, \\ \gamma_2 = \frac{g}{1-\alpha} - kr_1(1-\tau) \text{ if } s \geq q^{99}. \end{cases}$$

A steady state exists for a value $k > 1$ if we can solve simultaneously the equations (20), (21), and the equations (13) to (16) in section 2.2 that define K^*, r, w and s_1. Since r, w and s_1 are expressed as functions of K^*, and

$$K^* = \int \frac{(1-\tau)ws + \tau(w\bar{s} + rK^*) - c_0}{\gamma(s)} dF(s) \tag{22}$$

by integrating (21), this requires only solving the two simultaneous equations (20) and (22) for K^* and r_1.

[9] This assumption is in line with the empirical literature referred above, where differentiated rates of returns are significant especially for the top of the wealth distribution.

We know there is a solution when $k = 1$ (the previous section). What happens as k increases? Computation shows that a steady state exists for all values of $k > 1$. We do not prove this analytically, but demonstrate it, hopefully convincingly, by reporting the solution to equations (20) and (22) for many values of k in Table 4.

					Wealth shares			
k	r_1	kr_1	γ_1	γ_2	50th-90th	90th-99th	top 1%	top .1%
1	6.32%	6.32%	0.0089	0.0089	8.01%	66.63%	25.36%	4.51%
1.1	6.10%	6.71%	0.0103	0.0064	6.92%	57.52%	35.56%	6.32%
1.3	5.43%	7.06%	0.0147	0.0041	4.85%	40.36%	54.79%	9.74%
1.5	4.77%	7.16%	0.019	0.0035	3.76%	31.27%	64.97%	11.55%
2	3.58%	7.23%	0.027	0.0030	2.67%	22.21%	75.12%	13.35%
2.5	2.9%	7.25%	0.031	0.0029	2.29%	19.06%	78.65%	13.98%
3	2.42%	7.26%	0.034	0.0028	2.08%	17.28%	80.65%	14.32%
5	1.46%	7.27%	0.041	0.0027	1.76%	14.65%	83.59%	14.86%
10	0.73%	7.28%	0.045	0.0026	1.58%	13.12%	85.31%	15.16%

Table 4: Steady state allocation as a function of k

We do not report the values of r, K^*, w and s_1 in Table 4 because they are not affected by the value of k, and are thus the same as in the preceding section. We observe that r_1 decreases with k, and we now show that it converges to 0 as k becomes large. If this were not the case, since the left-hand side of equation (20) is constant as a function of k, it would follow that

$$\int_{q99}^{\infty} \frac{S^*(s)dF(s)}{1+g}$$

must tend to zero as k becomes large, which is clearly false from the definitions. Therefore r_1 must tend to zero, and we conjecture that

$$\lim_{k\to\infty} r_2 = \frac{g}{(1-\alpha)(1-\tau)} = 7.69\%,$$

which is the value that renders $\gamma_2 = 0$. In other words, γ_2 never becomes negative – for that would indicate the non-existence of a steady state.

Observe from Table 4 that r_1 plunges from 6.3% to 0.7% as k increases from 1 to 10, but that kr_1 does not increase very much, moving from 6.3% to 7.3%. Even though the absolute level of the return on wealth of the top

1% does not increase much, the wealth shares of various groups change a lot with k. The wealth shares of the top 1% and top 0.1% increase with k, at the expense of the shares of the patrimonial middle class and of the 90th to 99th centiles in the wealth distribution. It is striking that steady state wealth shares are very sensitive to k when k is low: an increase in k from 1 to 1.3 more than doubles the steady state wealth shares of both the top 1% and the top 0.1%! In other words, a rate premium of 30% for the top 1% results in more than half of total wealth being concentrated among them (up from a quarter with identical rates of return for all).

Since $r_1 \to 0$ as k becomes large, it follows that $\gamma_1 \to g/(1-\alpha)$, and from equation (22) that the wealth of the bottom 99% approaches

$$\int \frac{(1-\tau)ws + \tau(w\bar{s} + rK^*) - c_0}{g/(1-\alpha)} dF(s).$$

Dividing this number by K^* from Table 1, we conclude that, as $k \to \infty$, the wealth share of the top 1% approaches 86.7% . In Figure 7, we report calculation of the wealth share of the top 1% for steady states up to $k = 20$. The figure bears out our conjecture that the limiting value of the top wealth share is 86.7%.

Insert Figure 7: Wealth share of the top 1%, steady states for $3 \leq k \leq 20$

We now say a quick word about the convergence pattern to the steady state when one starts with the wealth distribution obtained from Saez and Zucman (2016), as in the preceding section. Fixing a value of $k > 1$, we do observe the same type of convergence as the one reported in section 2.4, namely a non-monotone convergence to their steady state values for the interest rate, the wage, capital stock, and some wealth shares. The conclusions obtained in this section regarding convergence then carry through to the introduction of differentiated rates of return.

We summarize the results of this section as follows. First, as the rate of return to wealths in the top 1% becomes an arbitrarily high multiple of the rate of return to wealths in the bottom 99%, steady states continue to exist, where incomes and wealths grow at the rate g. Second, the values of K^*, r, w and s_1 are independent of how capital income is distributed among owners of capital. Third, the wealth share of the top 1% increases very fast with k for low values of k, and approaches an asymptotic value less than one when k becomes very large. Fourth, the convergence pattern studied in section 2.4 carries through to the case of differentiated rates of return on wealth.

The fact that the fundamental values K^*, r, w and s, are independent of how returns to capital are distributed reflects a particular view about the role of the financial sector. Competitive forces determine the fundamentals of the economy, and the role of the financial sector, which expends great energy in seeking high returns for wealthy clients, is simply to allocate profits in favor of their clients. In our models, the financial sector is unproductive, as the well known quip by Paul Volcker asserts.

We now move to the introduction of social mobility.

4 Social mobility

4.1 Analytical model

We now introduce death and inheritance. When an adult dies, we assume that his capital passes (untaxed) to his only child. However, the child will not in general have the skill/income capacity of the father. We use the 100X100 intergenerational income mobility matrix of Chetty *et al* (2014) to model this process.[10] An element p_{ij} of this matrix is the fraction of sons of fathers at the i^{th} centile of the income distribution who have incomes at the j^{th} centile of their cohort's income distribution. Indeed, we assume that the matrix $P = \{p_{ij}\}$ defines the mobility of skill, hence earned income.

To describe the dynamics, let us first suppose that all fathers die at once at the beginning of the year. If an s father dies at the beginning of year t, his son inherits $S_{t-1}(s)$. The son will be economically active beginning in year t. The sons are distributed on the skill distribution F according to P. Denote by Q^i the i^{th} centile of F, comprising a small interval of skills. Let $s \in Q^i$ and $s' \in Q^j$. Then the 'number' of sons who inherit from fathers of skill s and end up at skill level s' will be equal to $100 p_{ij} f(s) f(s')$. Integrating this

[10] Online Data Table 1: National 100 by 100 transition matrix, available online at http://obs.rc.fas.harvard.edu/chetty/website/v2.1/online_data_tables.xls. We use the transpose of the matrix therein.

over $s' \in Q^j$ gives

$$\int_{Q^j} 100 p_{ij} f(s) f(s') ds' = p_{ij} f(s),$$

because

$$\int_{Q^j} f(s') ds = \frac{1}{100},$$

which is the correct number of sons of s fathers who end up at Q^j. If we add up these numbers over all j, we have

$$\sum_j p_{ij} f(s) = f(s),$$

which is the total number of sons whose fathers were of skill s.

Now let's look at all fathers $s \in Q^i$. The number of sons who end up at some $s' \in Q^j$ will be

$$\int_{Q^i} 100 p_{ij} f(s') f(s) ds = p_{ij} f(s').$$

Summing over i, we have

$$\sum_j p_{ij} f(s') = f(s'),$$

which is the correct number of sons at s'.

We now compute the total wealth inherited by children when their fathers' estates pass to them. We are interested in the total savings at a generic skill level s' in the son's generation. These savings come from fathers' wealth at date $t-1$, so these are the savings at the beginning of date t for the sons, before they have augmented their savings with their own income (which will be with wages at skill level s'). The amount of inheritance at each value of father's s will be the same. The average inheritance for sons at s' in centile j will be

$$\sum_{i=1}^{100} 100 p_{ij} \int_{Q^i} S_{t-1}(s) f(s) ds \equiv \tilde{S}_{t-1}(s'), \text{ for } s' \in Q^j, \ j = 1, ..., 100. \qquad (23)$$

We now drop the assumption that all fathers die at the beginning of the year, and suppose, instead, that a fraction q of all fathers die at each skill level at the beginning of each year. Thus the capital at end of date t at skill level s will be an average of the capital of those who did not die and of the newly inheriting sons who arrive at skill level s. The average capital at skill level s at the end of the year is

$$S_t(s) = (1-q)S_{t-1}(s) + q\tilde{S}_{t-1}(s) + I_t(s). \tag{24}$$

The quantity in the first part of this convex combination is the average wealth of those who do not die at the beginning of date t– call it 'survivors' capital'– and $\tilde{S}_{t-1}(s)$ is the average wealth of sons who join type s at the beginning of date t.

However, we do not attempt to keep track of the heterogeneity of wealth at each skill level that occurs as a result of death and inheritance. We only track average wealth at each skill level. Thus, we aggregate at each skill level s at each date, and assign everyone of that skill level the average amount of capital from (24). If you are the son of a wealthy father, your inheritance will add wealth to the cohort at your skill level, but it will not benefit you especially.

The equations summarizing the model are the same equations given at the end of section 2.1, except that we substitute equation (24) for (4) for the intergenerational transmission of wealth, making use of equation (23) defining the wealth of an inheriting son, at the beginning of date t, whose own skill level is s'.

As for solving the model, we proceed as previously up to and including the solving of consumption and investment. $\tilde{S}_{t-1}(.)$ is then determined by (23), and finally, $S_t(.)$ is determined by (24).

4.2 Numerical Results: Steady state and convergence

We now simulate the model, beginning with the Saez-Zucman (2016) wealth function for the US in 2012 as $S_0(s)$. We let 2% of the population die each year (thus, $q = 0.02$). The simulation results are to be compared with those in Figures 1-6, pertaining to our vanilla model.

Add Figures 8 to 13 around here:

It appears as if the model approaches a steady state (see figures 8 to 13). It is not as easy as previously to prove this, and we defer to Annex 3 for the analytical definition of a steady state and its solution with social mobility. We report in Table 5 the steady state allocations with and without social mobility, the latter corresponding to section 2.2.

Steady state values	Without social mobility	With social mobility
r	6.32%	13.64%
w	\$67,750	\$45,334
K/y	4.45	2.60
Wealth shares		
bottom 50%	0%	0.54%
50th-90th	8%	0.86%
top 1%	25.4%	40.97%
top 0.1%	4.5%	8.49%

Table 5: Steady state allocation with and without social mobility

Perhaps the most significant result is that the capital-income ratio converges to a far smaller level than in the vanilla model: K/y appears to converge to around 2.6 as opposed to 4.45 in the vanilla model. The reason for this is quite clear. Many sons of wealthy fathers themselves have low earning power, and they consume their inheritance. Correspondingly, because of the scarcity of capital, the interest rate converges to a much higher number here, about 13.6%, than in the vanilla model. In this model, then, $r > g$ with a vengeance in the steady state. The relative paucity of labor means that the wage, normalized by the growth rate converges to \$45,300 per annum, far less than the wage in the vanilla model (normalized by the growth factor), which converges to \$67,700 per annum. Labor's income share also decreases compared to the basic version of the model. Thus, ironically, the churning of capital via inheritance renders the working class relatively worse off than without churning, because low-wage inheritors 'squander' their wealth, and capital becomes relatively scarce. We are reminded of Marglin's (1974) proposition that the social function of capitalists is to accumulate.

We next present in Table 5 the values to which the wealth shares of various quantiles appear to converge in the steady state. For the first time, the bottom half of the population accumulates some wealth: this is due to churning. They end up with about 0.5% of total wealth. The patrimonial middle class is immiserated: they end up after 300 years with 0.86% of the total wealth. As a consequence, the wealth share of the top decile converges to around 98% of the total wealth.

As for the convergence process to this steady state, Figures 8 to 13 show that it is non monotonic for several variables, with overshooting of the steady state values for several variables such as w, r and K/y. The non-monotonicity is especially visible for the wealth share of the top 1%. Even though we start from a value which is close to the steady state level, the wealth share decreases over 50 periods to 35%, and then increases towards the steady state level of 41%.

The impact of lowering c_0 from \$100,000 to \$50,000 is reported in Table 6 and is similar to what we obtained in the base case model. Halving the expected standard of living allows the middle class to start accumulating, resulting in a capital/output ratio increasing by a factor close to four. The rate of return on capital decreases tenfold to 1.3%, while the wage rate more than doubles to \$110,000. As for wealth shares, the bottom half accumulates more than a third of total wealth, and the patrimonial middle class close to 45%. The decrease in top wealth shares is especially impressive.

Steady state values	$c_0 = \$50,000$	$c_0 = \$100,000$
r	1.27%	13.64%
w	\$111,162	\$45,334
K/y	9.87	2.60
Wealth shares		
bottom 50%	37.14%	0.54%
50th-90th	44.91%	0.86%
top 1%	2.66%	40.97%
top 0.1%	0.36%	8.49%

Table 6: Steady state allocations with social mobility as a function of c_0

The society that we describe after 300 years of capital churning and a standard of living of \$100,000 is in all likelihood not one that we will ever see. First, it is hard to imagine that an economy would be sustainable in which over 95% of the wealth accrues to the top decile. Secondly, the wealth-income ratio, of below three, does not seem to be one that would evolve. We attribute these unrealistic results to the rather unrefined process of inheritance that we used. First of all, we have used the Chetty *et al* (2014) income to income intergenerational transition matrix (specified in terms of father's and son's ranks) rather than an earnings to earnings matrix, which is what we really require for our model. What we need is a mapping from father's labor earnings to son's labor earnings, which we do not have. Secondly, when a son inherits, his capital is shared by the entire earnings cohort to which he belongs. We adopted this procedure because it is far simpler than keeping track of individual inheritances, which would introduce heterogeneity of wealth among the members of each skill cohort. Thirdly, we have ignored assortative mating, which would considerably reduce capital churning. Repairing these short-cuts would, we presume, give us a more realistic picture of the effects of intergenerational mobility on the distribution of wealth.

5 Taxing capital

We finally study the effect of capital taxation by examining the steady states generated at different capital levies. Denote by τ_1 a per annum tax on the individual's wealth, collected at date t but upon wealth at date $t-1$. We choose to levy the tax only on the top decile of the distribution. We amend the vanilla model of section 2: thus, there is no intergenerational mobility, and there is one rate of return on capital. The revenues from the capital tax will be distributed as a demogrant to the entire population.

At the steady state, it will continue to be true that those who invest positively will be exactly those types for whom $y^*(s) > c_0$. We continue to denote by s_1 the largest type whose savings are zero in the steady state. Thus in the steady state, $S^*(s)$ remains given by equation (17) which can be expressed for $s \geq s_1$ as

$$\begin{aligned} S^*(s) = \frac{(1+g)(1-\alpha)}{g} & \left[(1-\tau)\left(ws + \frac{r}{1+g}S^*(s) \right) + \tau\left(w\bar{s} + rK^* \right) \right. \\ & \left. - \mathbf{1}_{[q90,\infty]}(s)\tau_1 \frac{S^*(s)}{1+g} + \tau_1 \Lambda^* - c_0 \right], \end{aligned} \tag{25}$$

where

$$\mathbf{1}_{[q90,\infty]}(s) = \begin{cases} 1, \text{ if } s \geq q90, \\ 0 \text{ otherwise,} \end{cases}$$

and

$$\Lambda^* = \int_{q90}^{\infty} \frac{S^*(s)}{1+g} dF(s)$$

is the steady-state value of capital held by the top decile, and where w, r, and K^* are steady-state values. In the income term in (25), we collect the capital tax if and only if $s \geq q90$, but everyone receives the demogrant $\tau_1 \Lambda^*$. Gathering together terms, we can rewrite (25) as

$$\frac{S^*(s)}{1+g} = \left(\frac{(1-\tau)ws + \tau w\bar{s} + \tau r K^* + \tau_1 \Lambda^* - c_0}{\frac{g}{1-\alpha} - r(1-\tau) + \mathbf{1}_{[q90,\infty]}(s)\tau_1} \right) \text{ for } s \geq s_1. \tag{26}$$

We continue to have the market-clearing equation for capital

$$K^* = \int_{s_1}^{\infty} \frac{S^*(s)}{1+g} dF(s). \tag{27}$$

The upper bound of types who save zero in steady state is defined by

$$s_1 = \frac{c_0 - \tau\left(w\bar{s} + rK^*\right) - \tau_1 \Lambda^*}{(1-\tau)w}.$$

We integrate equation (26) over the interval $[s_1, \infty)$. The left-hand side becomes K^* by equation (27). This new equation contains two unknowns, K^* and Λ^*: note that w, r, and s_1 are all functions of K^* and Λ^*. Secondly, we integrate equation (26) over the interval $[q90, \infty)$: then the left-hand side of the new equation is Λ^*, and thus we have a second equation in K^* and Λ^*. We now solve these two equations simultaneously for K^* and Λ^*, for various values of the capital tax τ_1. A solution is the steady-state that we seek.[11]

We calculated steady states for values of the capital levy between 0 and 3%. In this interval, the bottom half of the population continues to accumulate zero wealth: they use the capital levy demogrant to augment consumption. However, the fortunes of the patrimonial middle class, from the 50th to 90th centile, improve dramatically. In Figure 14, we plot the wealth shares of three quantile groups of the population at the steady state, as a function of the capital levy. With an increase of the wealth tax from zero to 3%, the wealth share of the middle class increases from below 10% to almost 70%; obviously, this is at the expense of the top decile.

[11]The procedure described is the correct one as long as $s_1 \leq q90$. This turns out to be the case in the region that we examine. If $s_1 > q90$, a slightly different procedure must be used.

Insert Figures 14 to 18 around here

In figures 15, 16 and 17, we plot the interest rate, capital-output ratio and labor's share in national income as a function of the capital tax. It appears that most of the action occurs as the tax increases from zero to 1%. These variables are all quite stable in the higher part of the range, although the wealth shares continue to change quite dramatically.

The lesson seems to be that quite moderate capital taxation (vastly short of full appropriation of capital) has a dramatic effect on the fortunes of the middle class, but no effect on the wealth of the bottom half, who continue to own nothing. Their consumption, however, increases due to the capital levy demogrant. Evidently, other strategies must be used to create wealth for the bottom half of the income distribution.

What is the relative size of the demogrant from taxing capital income, which is $\tau r K^*$, and from the capital levy, which is $\tau_1 \Lambda^*$? We plot these two demogrants in figure 18. We see that at a 3% capital levy, the capital-stock demogrant is only about one-third the value of the capital-income demogrant. As Piketty (2014) emphasizes, a principal value to having a capital stock tax, even a small one, is that it would establish statistics on the distribution of the wealth, which would presumably invigorate the social movement for redistribution.

6 Conclusion

We close by drawing attention to some key points. We believe modeling consumers as seeking to reach a socially acceptable and culturally determined level of consumption, which we denoted c_0, and beyond that accumulating wealth for its own sake, in addition to augmenting consumption, is a good approximation to reality in a capitalist society in which high consumption and accumulation of wealth are prized as signals of success. In the vanilla model (section 2) and its three variants, steady states always exist. Although the top quantiles of the distribution can own, in the limit, large fractions of total wealth, these steady-state values are less than unity. When there are differential rates of return to capital, depending on the size of the investment, even if the top 1% receive a rate of return that is an arbitrarily high factor of that received by the bottom 99%, the top 1% does not in the limit own all the capital.

Not only do steady states exist in all our variants, but it appears that convergence to the steady state occurs from an arbitrary initial distribution of wealth following our dynamics. This is a conjecture, stimulated by our simulations. A proof would presumably show that the mapping of the wealth function at date t to date $t + 1$ is a contraction mapping. We illustrate convergence to the steady state from the 2012 wealth distribution of Saez and Zucman (2016). Two features are noteworthy: convergence takes a very long time, and in several variables of interest it is not monotonic. This non-monotonicity illustrates another theme from Piketty (2014): that it is imprudent to attempt to deduce general dynamical laws of capitalism from time series that are short.

Intergenerational mobility provides some wealth to the bottom half, but it also increases the wealth concentration at the top. This is apparently due to the fact that when wealth travels down the distribution through low-earning children inheriting from wealthy fathers, it is largely consumed rather than saved. A capital tax levied on the top decile at a modest rate (below 3%), and redistributed as a demogrant, has a very dramatic effect on the wealth share owned by the middle class, which increases from 10% to 70%. It has no effect on the wealth held by the bottom half, who consume the demogrant in an attempt to reach the consumption level.

Finally, the only variant we studied that succeeds in creating wealth for the bottom half of the distribution is a reduction in the level of c_0. If c_0 is halved, from \$100,000 to \$50,000 per annum, the bottom 90% of the distribution converge to owning 75% of the wealth – almost their per capita share. Aside from the fact that austerity of this kind would have massive negative effects on employment during the transition to a new equilibrium, we do not believe the strategy is psychologically feasible in the US. We conjecture that a lower c_0 may correspond to the social compact in Europe. If that is so, we can predict a quite different long-run distribution of wealth in Europe, and a higher capital/output ratio in Europe than in the US.

Annex 1: Calibration of the CES production function

From (7), we obtain

$$\begin{aligned} \frac{(r+d)K}{y} &= a\left(\frac{y}{AK}\right)^{\frac{1-\delta}{\delta}} \\ \Leftrightarrow a &= \left(\frac{rK}{y} + d\frac{K}{y}\right)\left(\frac{y}{AK}\right)^{\frac{\delta-1}{\delta}}. \end{aligned} \tag{28}$$

Taking $K/y = 4.5$ and capital's share in income $rK/y = 0.28$ (both from Piketty (2014)), (28) reduces to

$$a = (0.28 + 0.09)(4.5)^{0.176}A^{0.176}. \tag{29}$$

From the production function, we have

$$1 = A\left(a(\frac{K}{y})^{-0.176} + (1-a)(\frac{L}{y})^{-0.176}\right)^{-5.67}. \tag{30}$$

We calculate

$$\frac{L}{y} = (108.3)^{-1},$$

using the facts that total income is $16.8X10^9$ in thousands of dollars and the size of the labor force is $155X10^6$ (so that income per worker is \$108,300). Solving (29) and (30) simultaneously, we obtain (a, A)=(0.636,4.804).

Annex 2: Impact of elasticity of substitution on the steady state in the vanilla model

					Wealth shares			
δ	w	r	K/y	$w\bar{s}/y$	0-50	50-90	90-99	top 1%
0.85	\$67,750	6.321%	4.454	0.629	0	8.01%	66.63%	25.36%
0.95	\$67,760	6.320%	4.450	0.630	0	8.00%	66.63%	25.37%
1.05	\$67,770	6.317%	4.446	0.630	0	8.00%	66.63%	25.37%
1.15	\$67,779	6.316%	4.442	0.631	0	7.99%	66.64%	25.37%
1.25	\$67,788	6.314%	4.438	0.631	0	7.98%	66.64%	25.38%

Table A2.1: Steady state results in vanilla model as a function of δ

Annex 3: Computation of the steady state with intergenerational mobility

We solve analytically for the steady state with intergenerational mobility, which according to simulations, appears to exist. Substituting equation (23) into (24) we have, for any s,

$$S_t(s) = (1-q)S_{t-1}(s) + q\sum_{i=1}^{100} 100 \int\limits_{\sigma \in C^{-1}(i)} p(i, C(s))S_{t-1}(\sigma)f(\sigma)d\sigma + I_t(s). \quad (31)$$

We have introduced some new notation in (31) for clarification. $C(s)$ is the quantile of distribution F to which s belongs. Consequently, $C^{-1}(i)$ is the set of workers whose skills are in the i^{th} quantile of F. If there is a steady state, then the steady-state savings function and investment function satisfy

$$S^*(s) = \frac{1-q}{1+g}S^*(s) + q\sum_{i=1}^{100} 100 \int\limits_{\sigma \in C^{-1}(i)} p(i, C(s))\frac{S^*(\sigma)}{1+g}dF(\sigma) + I^*(s). \quad (32)$$

Denote

$$\beta_j = \int\limits_{s \in C^{-1}(j)} S^*(s)dF(s),$$

and integrate (32) over $s \in C^{-1}(j)$ to obtain

$$\beta_j = \frac{1-q}{1+g}\beta_j + q\sum_{i=1}^{100} p(i,j)\frac{\beta_i}{1+g} + \int\limits_{\sigma \in C^{-1}(j)} I^*(\sigma)dF(\sigma). \quad (33)$$

The difficult part is that the last term on the r.h.s. of (33) will have a zero value for all quantiles j who eat up their entire savings each period (these will be low-skilled sons who recently inherited, but not enough to reach the required consumption of c_0) and it will have a positive value for those who have a lot of savings or high earnings. We have no way of guessing at what quantile j this break occurs. (And the break itself is approximate – it will almost surely be the case that some part of the critical quantile will actually invest positively, and the other part will not.) Forgetting this last complication, we can say that we will have, for some quantile j^*:

$$0 < j < j^* \Rightarrow \qquad \int\limits_{s \in C^{-1}(j)} I^*(s) dF(s) = \frac{-\beta_j}{1+g}, \tag{34}$$

$$\begin{aligned} j^* \leq j \leq 100 \Rightarrow \qquad & \int\limits_{s \in C^{-1}(j)} I^*(s) dF(s) \\ = \; & (1-\alpha)\left((1-\tau)\left(ws^j + r\frac{\beta_j}{1+g}\right) + \frac{\tau\left(w\bar{s} + rK^*\right) - c_0}{100}\right), \end{aligned} \tag{35}$$

where

$$s^j = \int\limits_{s \in C(j)} s dF(s).$$

Our procedure is to *guess* the value of j^* by examining our 300 period simulation result. We see there that at the end of that simulation, the only quantiles that are consuming more than c_0, normalized by the growth factor $(1+g)^{299}$, are centiles 92 through 100.[12] Thus, in the steady state, the first 91 centiles are consuming their entire savings at each date, following our consumption rule.

We now insert (34) into equation (33), and, after a little re-arranging to combine like terms in the β's, write these equations as

$$0 < j < 91 \Rightarrow \qquad \frac{1 + g + q\left(1 - p(j,j)\right)}{1+g}\beta_j - q\sum_{i \neq j}^{100} p(i,j)\frac{\beta_i}{1+g} = 0, \tag{36}$$

$$\begin{aligned} 92 \leq j \leq 100 \Rightarrow \qquad & \frac{g + q\left(1 - p(j,j)\right) - (1-\alpha)(1-\tau)r}{1+g}\beta_j - q\sum_{i \neq j}^{100} p(i,j)\frac{\beta_i}{1+g} \\ = \; & (1-\alpha)\left((1-\tau)ws^j + \frac{\tau\left(w\bar{s} + rK^*\right) - c_0}{100}\right). \end{aligned} \tag{37}$$

System (36)-(37) comprises 100 linear equations in the $\{\beta_j\}$. There are in addition the unknowns r, w and K^*, but r and w are functions of K^* via

[12] This does not contradict the observation made in section 4.2 that the bottom 90% accumulates some wealth at the steady state. This wealth is indeed inherited from fathers, but eaten up by sons in their attempt to reach the standard of living c_0.

equations (14) and (15). The 101st equation is

$$K^* = \sum_{j=1}^{100} \frac{\beta_j}{100}.$$

A solution to these equations is a steady state of the model. We cannot expect to have a perfect solution, because of the problem mentioned above equation (34): that some members of the critical quantile j^* will be negative savers and some positive savers. This could only be addressed by having a continuous transition matrix.

We attempted to verify that our simulations produced an approximate steady state as follows. We took the savings function at the 300th iteration of the program, and computed the 100 dimensional vector β using it, and also the values r, w and K^*. We then computed the 100 linear equations (36)-(37). The error is small. Instead of the vector zero (of dimension 100) we got a vector of length 0.39. The average length of the deviation from the zero vector along any dimension is 0.012.

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Figure 1: Basic model, convergence of r

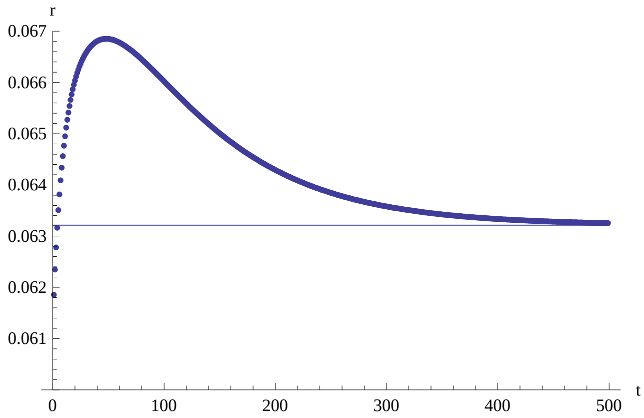

Figure 2: Basic model, convergence of w

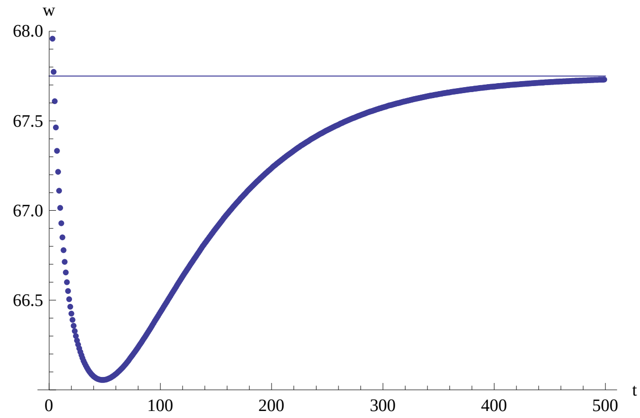

Figure 3: Basic model, convergence of capital/output ratio

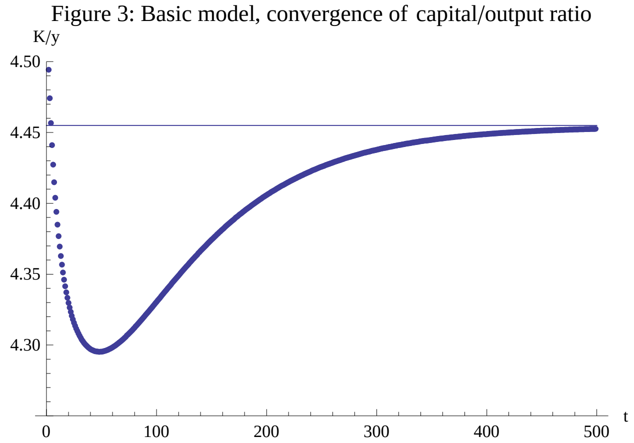

Figure 4: Basic model, convergence of patrimonial middle class wealth share

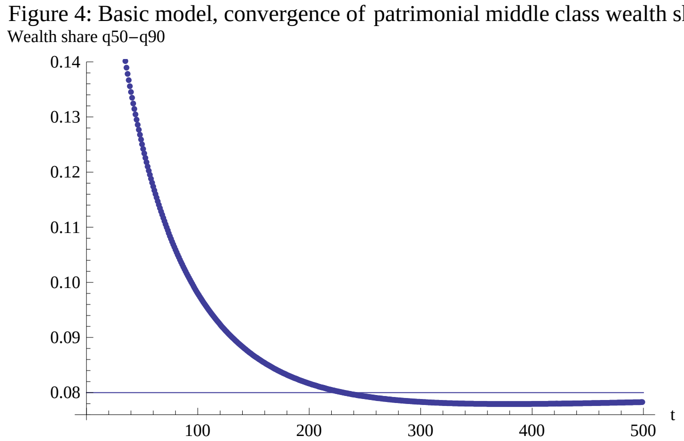

Figure 5: Basic model, convergence of top 1% wealth share

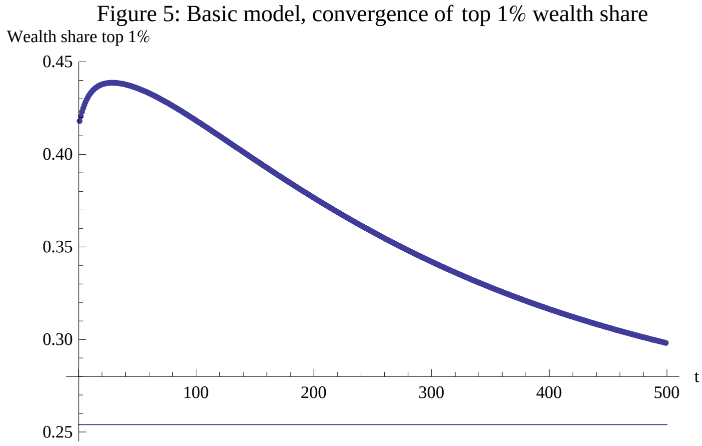

Figure 6: Basic model, convergence of top 0.1% wealth share

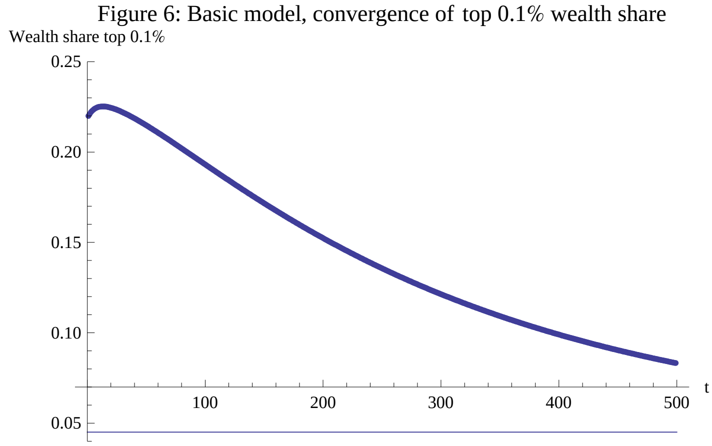

Figure 7: Top 1% wealth share as a function of k

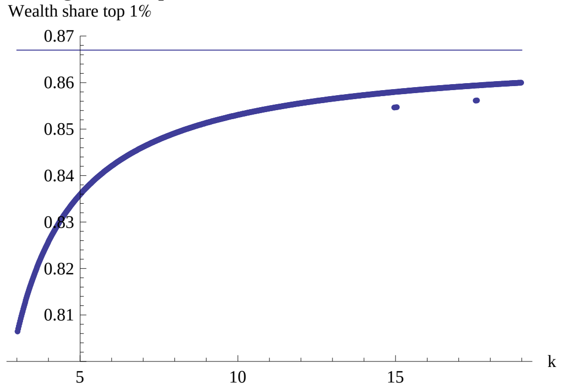

Figure 8: Convergence of r with social mobility

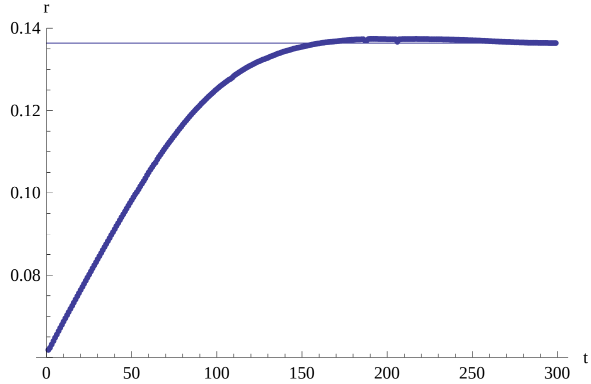

Figure 9: Convergence of w with social mobility

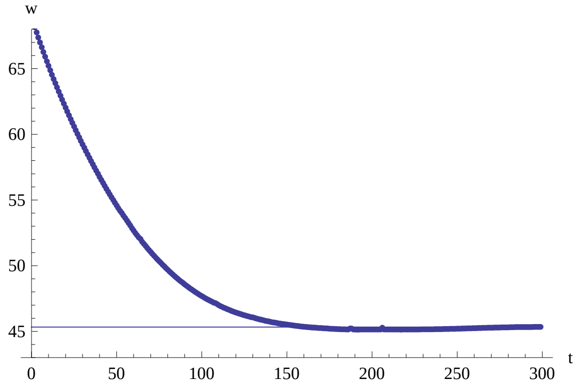

Figure 10: Convergence of capital/output ratio with social mobility

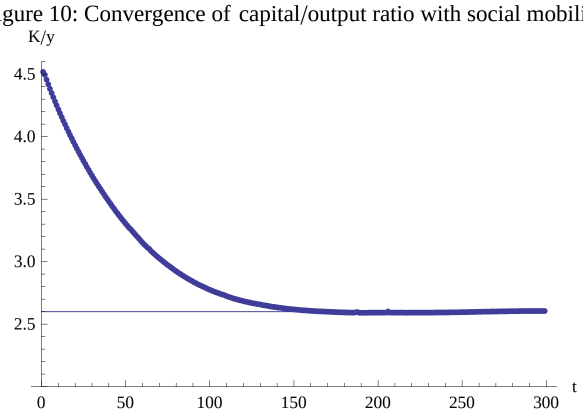

Figure 11: q50 to q90 wealth share with social mobility

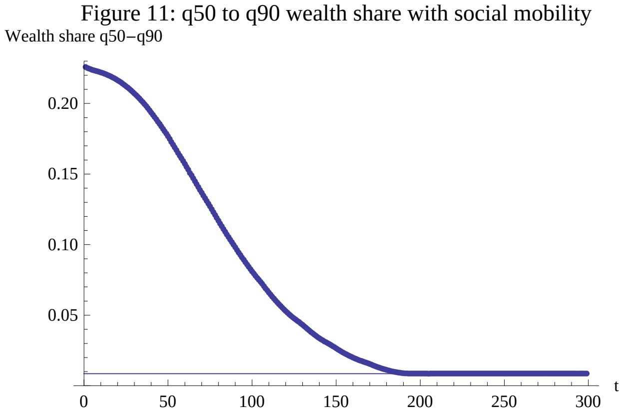

Figure 12: q90 to q99 wealth share with social mobility

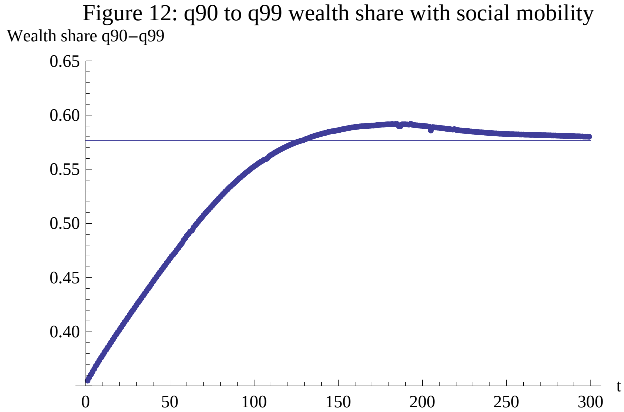

Figure 13: Top 1% wealth share with social mobility

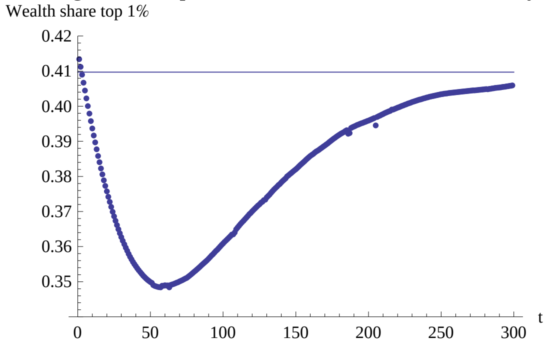

Figure 15: Interest rate as function of capital levy

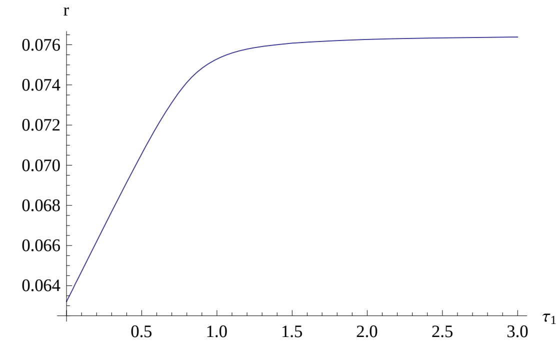

Figure 16: Capital−output ratio as function of capital levy

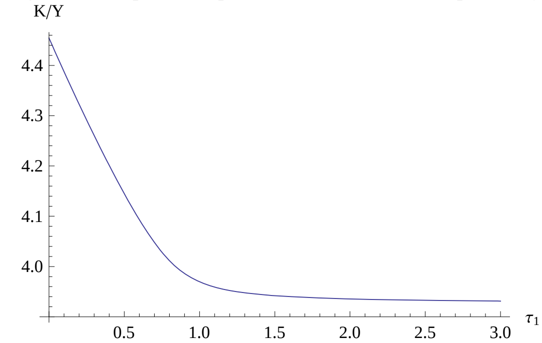

Figure 17: Labor share as a function of capital levy

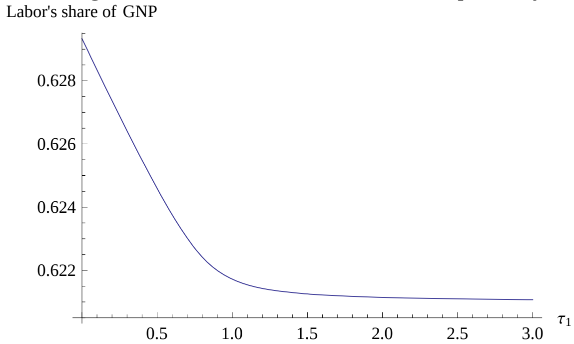

Figure 14: Quantile wealth shares as function of capital levy

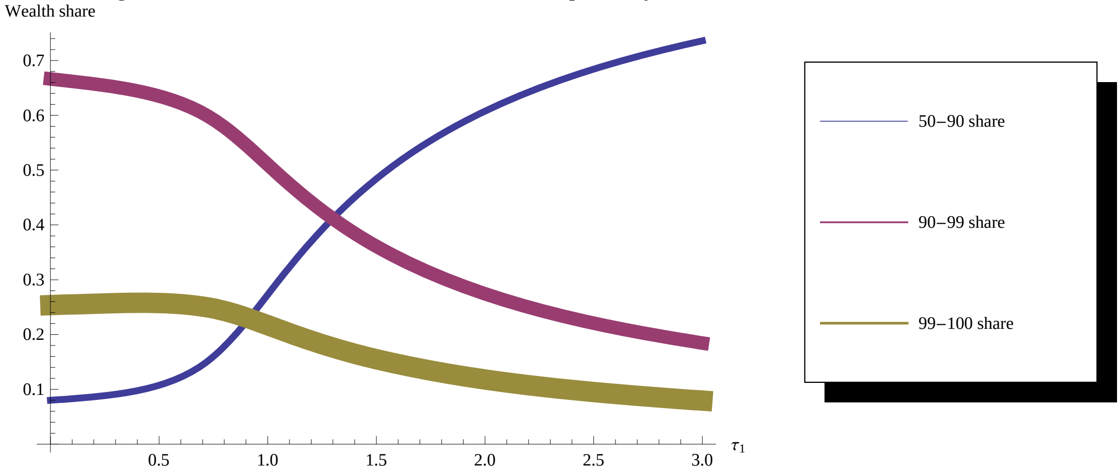

Figure 18: Demogrants as a function of capital levy